قاعدة لوبيتال L'Hopital's rule

اذا كان $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ أو $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \infty$ فان :

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

ملاحظة : تُستخدم قاعدة لوبيتال عند الحصول على

$\frac{0}{0}$ ، $\frac{\infty}{\infty}$ ، $\frac{0}{\infty}$ ، $\frac{\infty}{0}$ ، $0.\infty$ و $\infty.0$

مثال جد قيمة الغايات التالية :

1. $\lim_{x\to 0}\frac{e^{2x}-1}{x} = \lim_{x\to 0}\frac{2e^{2x}}{1} = 2$

2. $\lim_{x\to\infty}\frac{3x^2-x+5}{5x^2+6x-5} = \lim_{x\to\infty}\frac{6x-1}{10x+6} = \lim_{x\to\infty}\frac{6}{10} = \frac{3}{5}$

3. $\lim_{x\to\infty} x^2e^{-2x}$

$$= \lim_{x\to\infty}\frac{x^2}{e^{2x}} = \lim_{x\to\infty}\frac{2x}{2e^{2x}} = \lim_{x\to\infty}\frac{2}{4e^{2x}} = 0$$

4. $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

$$= \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} = \frac{1}{6}$$

5. $\lim_{x\to -1}\frac{1+\cos\pi x}{x^2+2x+1}$

$$= \lim_{x\to -1}\frac{-\pi\sin\pi x}{2x+2} = \lim_{x\to -1}\frac{-\pi^2\cos\pi x}{2} = \frac{\pi^2}{2}$$

6. $\lim_{x\to 0} x\csc^2\sqrt{2x} = \lim_{x\to 0}\frac{x}{\sin^2\sqrt{2x}}$

$$= \lim_{x\to 0}\frac{x}{\frac{1-\cos 2\sqrt{2x}}{2}} = \lim_{x\to 0}\frac{2x}{1-\cos 2\sqrt{2x}}$$

$$= \lim_{x\to 0}\frac{2}{2\sin 2\sqrt{2x}\times\frac{2}{\sqrt{2x}}} = \frac{1}{4}\lim_{x\to 0}\frac{2\sqrt{2x}}{\sin 2\sqrt{2x}} = \frac{1}{4}$$

7. $\lim_{x\to 0^+} \frac{\ln(\cos 3x)}{\ln(\cos 2x)}$

$$= \lim_{x\to 0^+} \frac{\frac{-3\sin 3x}{\cos 3x}}{\frac{-2\sin 2x}{\cos 2x}}$$

$$= \lim_{x\to 0^+} \frac{-3\tan 3x}{-2\tan 2x}$$

$$= \lim_{x\to 0^+} \frac{9\sec^2 3x}{4\sec^2 2x} = \frac{9}{4}$$

تمارين

جد قيمة ما يلي

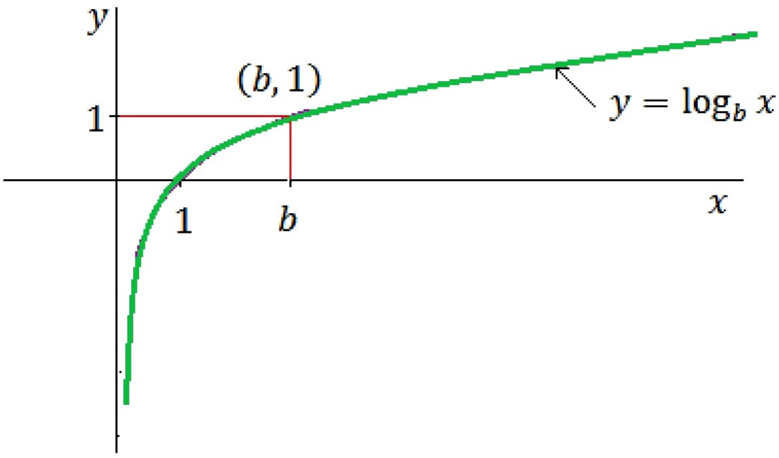
1. $\lim_{x\to 0} \frac{1-\cos x}{x^2+x}$
2. $\lim_{x\to 0} \frac{5-5\cos x}{e^x - x - 1}$
3. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} - x + 3}$
4. $\lim_{x\to 0} \frac{x - \ln(2x+1)}{x^2}$
5. $\lim_{x\to 0} (\csc x - \cot x)$
6. $\lim_{x\to 0} \frac{\tan x}{x + \sin x}$

Logarithm functions

The logarithm function with base b is the function $y = \log_b x$ where $b > 0$ and

The function is defined for all $x > 0$. $b \neq 1$.

Here is its graph for any base b.



Note the following:

1. For any base, the x-intercept is 1. ⇨ $\log_b 1 = 0$.

2. The graph passes through the point $(b, 1)$. ⇨ $\log_b b = 1$.

3. The graph is below the x-axis -- the logarithm is negative -- for $0 < x < 1$

4. The function is defined only for positive values of x.

5. The range of the function is all real numbers.

6. The negative y-axis is a vertical asymptote.

7. $\log_b(xy) = \log_b x + \log_b y$.

8. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.

9. $\log_b\left(\frac{1}{x}\right) = -\log_b x$.

10. $\log_b x^y = y \log_b x$.

11. For each strictly positive real number a and b, different from 1, we have

$$\log_b x = \frac{\log_a x}{\log_a b}$$

The natural logarithm $y = \ln x$

The system of natural logarithms has the number called e as its base. e is an irrational number; its decimal value is approximately 2.71828182845904.

To indicate the natural logarithm of a number we write "ln." $\ln x$ means $\log_e x$.

So we have

1. $\ln e = 1$
2. $\log_b x = \dfrac{\ln x}{\ln b}$
3. $\ln (xy) = \ln x + \ln y$
4. $\ln \left(\dfrac{x}{y}\right) = \ln x - \ln y$
5. $\ln x^n = n \ln x$

Derivative of natural logarithm function

If u is a function x, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}$$

Example 1: Find derivatives of the functions

1. $y = \ln(5x+1)$

$$\frac{dy}{dx} = \frac{1}{5x+1} \times 5 = \frac{5}{5x+1}$$

2. $y = 2x\tan^{-1} x - \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1} x - \frac{2x}{x^2+1} = 2\tan^{-1} x$$

3. $y = \ln(\sin 3x)$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times 3\cos 3x = 3\cot 3x$$

4. $y = \ln(x^2+3)^{(x^2+3)} \quad \Rightarrow y = (x^2+3)\ln(x^2+3)$

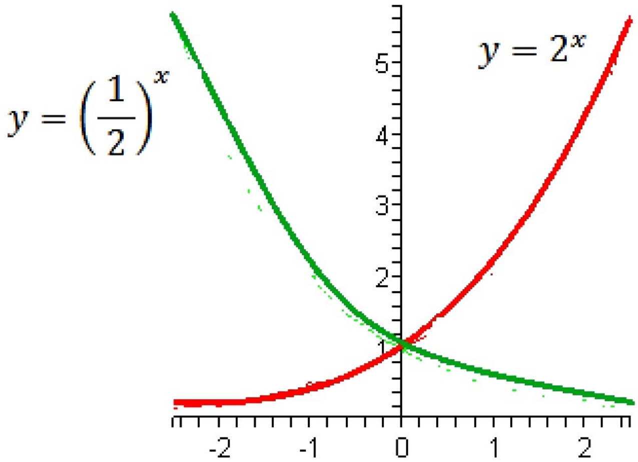
$$\frac{dy}{dx} = (x^2+3) \times \frac{2x}{(x^2+3)} + 2x\ln(x^2+3) = 2x + 2x\ln(x^2+3)$$

Exponential functions

For any positive number $a > 0 , a \neq 1$, there is a function called an exponential function that is defined as $f(x) = a^x$

For example

$$y = 2^x \;,\; y = \left(\frac{1}{2}\right)^x$$



Now, let's talk about some of the properties of exponential functions.

1. The graph of $f(x) = a^x$ will always contain the point$(0,1)$. Or put another way, $a^0 = 1$ regardless of the value of a.
2. For every possible a, $a^x > 0$. Note that this implies that $a^x \neq 0$.
3. If $0 < a < 1$ then the graph of a^x will decrease as we move from left to right.
4. If $a > 1$ then the graph of a^x will increase as we move from left to right.
5. If $a^x = b^x$ then $a = b$.

Basic rules for exponents

1. The product rule $\quad a^x . a^y = a^{x+y}$

2. The quotient rule $\quad \dfrac{a^x}{a^y} = a^{x-y}$

3. The rule for power of a power $\quad (a^x)^y = a^{x.y}$

Natural exponential function

The function $f(x) = e^x$ is often called exponential function or natural exponential function which is an important function. The exponential function $f(x) = e^x$ is the inverse of the logarithm function $f(x) = \ln x$.

Derivatives of exponential function

If u is a function x, then

$$\frac{d}{dx}(e^u) = e^u . \frac{du}{dx}$$

Example 2: Find y' and y'' of the functions

1. $y = e^{3x-2}$

$$y' = e^{3x-2} \times 3 = 3e^{3x-2}$$

$$y'' = 3e^{3x-2} \times 3 = 9e^{3x-2}$$

2. $y = 2xe^{1-5x}$

$$y' = 2xe^{1-5x} \times (-5) + 2e^{1-5x} = e^{1-5x}(-10x + 2)$$

$$y'' = e^{1-5x} \times (-10) + (-10x + 2)e^{1-5x} \times (-5)$$

$$= -5e^{1-5x}(2 - 10x + 2) = 10e^{1-5x}(5x - 2)$$

3. $y = e^{-3x} \sin 2x$

$$y' = 2e^{-3x} \cos 2x - 3e^{-3x} \sin 2x = e^{-3x}(\cos 2x - 3 \sin 2x)$$

$$y'' = e^{-3x}(-4 \sin 2x - 6 \cos 2x) - 3e^{-3x}(\cos 2x - 3 \sin 2x)$$

$$y'' = (5 \sin 2x - 2 \cos 2x)e^{-3x}$$

4. $y = e^{\sqrt{1-2x}}$

$$y' = \frac{-e^{\sqrt{1-2x}}}{\sqrt{1-2x}}$$

$$y'' = \frac{\sqrt{1-2x} \times \frac{e^{\sqrt{1-2x}}}{\sqrt{1-2x}} - \left(-e^{\sqrt{1-2x}}\right) \times \frac{-1}{\sqrt{1-2x}}}{1-2x} = \frac{e^{\sqrt{1-2x}} - \frac{e^{\sqrt{1-2x}}}{\sqrt{1-2x}}}{1-2x}$$

$$= \frac{e^{\sqrt{1-2x}}\left(1 - \frac{1}{\sqrt{1-2x}}\right)}{1-2x} = \frac{e^{\sqrt{1-2x}}\left(\frac{\sqrt{1-2x} - 1}{\sqrt{1-2x}}\right)}{1-2x} = \frac{e^{\sqrt{1-2x}}\left(\sqrt{1-2x} - 1\right)}{(1-2x)^{3/2}}$$

Solving exponential and logarithm equations

Logarithms are the "opposite" of exponentials. In practical terms, I have found it useful to think of logarithm in terms of the relationship:

$$y = \log_b x \quad \Leftrightarrow \quad x = b^y$$

$$y = \ln x \quad \Leftrightarrow \quad x = e^y$$

Chemistry application:

1- Radioactive Decay: The amount of a radioactive element A at time t is given by:

$$A = A_0 e^{kt}$$

Where A_0 is the initial amount of the element and k is the constant of proportionality.

Example 3: The radioactive element radium-226 has a half-life of 1620 years. If a sample initially contains 120 gm, find the constant k.

$$t = 1620 \quad \Rightarrow \quad A = \frac{1}{2}A_0 \quad \Rightarrow \quad A = \frac{1}{2} \times 120 = 60 \text{ gm}$$

$$A = A_0 e^{kt} \quad \Rightarrow \quad 60 = 120\, e^{k\times 1620}$$

$$e^{1620k} = \frac{60}{120} = 0.5$$

$$1620k = \ln(0.5)$$

$$k = \frac{\ln(0.5)}{1620} = \frac{-0.6931}{1620} = -4.28 \times 10^{-4}$$

Example 4: The radioactive element Iodine-131 has a half-life of 8 days. If a sample initially contains 5 gm . Find a function which gives the amount at any time t.

$$t = 8 \quad \Rightarrow \quad A = \frac{1}{2}A_0 \quad \Rightarrow \quad A = \frac{1}{2} \times 5 = 2.5 \text{ gm}$$

$$A = A_0 e^{kt} \quad \Rightarrow \quad 2.5 = 5\, e^{k\times 8}$$

$$e^{8k} = 0.5 \quad \Rightarrow \quad k = \frac{\ln(0.5)}{8} = -0.0866$$

$$\text{So} \quad A = A_0 e^{-0.0866\, t}$$

2- **The pH Scale**

pH is the negative logarithm of the hydrogen ion concentration:

$\mathrm{pH} = -\log[\mathrm{H}^+]$

Example 5: What is the pH of a solution where $[\mathrm{H}^+] = 0.0003\ M$.

$\mathrm{pH} = -\log[\mathrm{H}^+]$

$\mathrm{pH} = -\log(0.0003) = -\log(3 \times 10^{-4})$

$\mathrm{pH} = -(\log 3 + \log 10^{-4}) = -(\log 3 - 4\log 10)$

$\mathrm{pH} = -(0.447 - 4) = 3.553$

Example 6: How many moles of H^+ ions are in 1500 liters of human blood?. If $\mathrm{pH} = 7.4$

$\mathrm{pH} = -\log[\mathrm{H}^+]$

$7.4 = -\log[\mathrm{H}^+]$

$\log[\mathrm{H}^+] = -7.4$

$[\mathrm{H}^+] = 10^{-7.4} = 10^{0.6-8} = 10^{0.6} \times 10^{-8}$

$[\mathrm{H}^+] = 3.98 \times 10^{-8}\ M$

The moles of H^+ ions are in $1500 = 3.98 \times 10^{-8} \times 1500 = 5.97 \times 10^{-5}$

Exercises

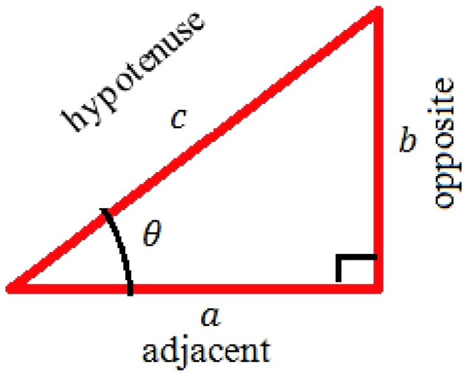
Find derivative in each of the following problems$(1-4)$

1. $y = \ln(x^2 + x)$
2. $y = x^3 \ln(x^2 - 2x + 5)$
3. $y = e^{\sin^{-1} x}$
4. $y = x^3 e^{-5x}$
5. The radioactive element Chromium- 51, has a half-life of 27.7 days. If a sample initially contains 75 milligrams. Find a function which gives the amount at any time t.
6. What is the pH of a solution where $[\mathrm{H}^+] = 0.000014\ M$.
7. Find $[\mathrm{H}^+]$ if $\mathrm{pH} = 8.5$

Trigonometric functions

Definitions of trigonometric functions for a right triangle

A right triangle is a triangle with a right angle (90°)



For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2 + b^2 = c^2$.

For angle θ, the trigonometric functions are defined as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c} \quad , \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{adj}} = \frac{b}{a} \quad , \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a} \quad , \quad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

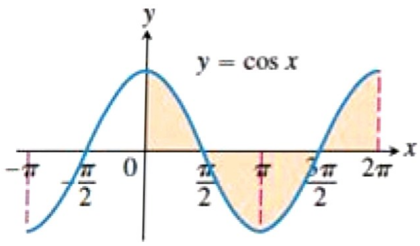
Trigonometric functions of negative angles

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$

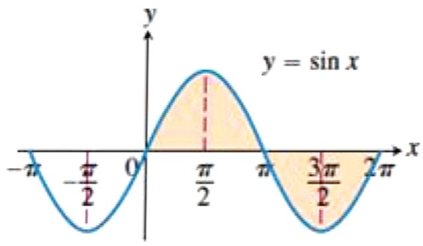
Some useful relationships among trigonometric functions

1. $\sin^2 x + \cos^2 x = 1$, $\sec^2 x - \tan^2 x = 1$, $\csc^2 x - \cot^2 x = 1$
2. $\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
3. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$, $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

Graphs of Trigonometric Functions



Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



Domain: $-\infty < x < \infty$
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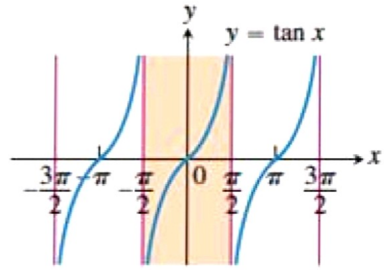
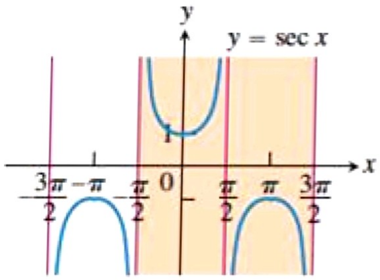
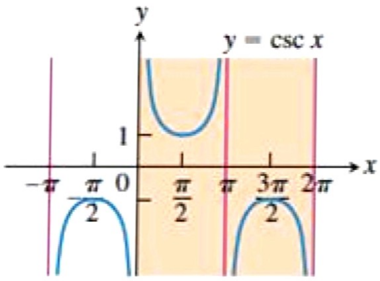
Period: 2π



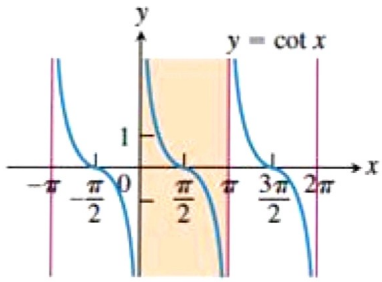
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

Derivatives of trigonometric functions

If u is a function x, the chain rule version of this differentiation rule is

1. $\frac{d}{dx}(\sin u) = \cos u \,.\, \frac{du}{dx}$

2. $\frac{d}{dx}(\cos u) = -\sin u \,.\, \frac{du}{dx}$

3. $\frac{d}{dx}(\tan u) = \sec^2 u . \frac{du}{dx}$

4. $\frac{d}{dx}(\cot u) = -\csc^2 u . \frac{du}{dx}$

5. $\frac{d}{dx}(\sec u) = \sec u \tan u \,.\, \frac{du}{dx}$

6. $\frac{d}{dx}(\csc u) = -\csc u \cot u \,.\, \frac{du}{dx}$

Example 1: Find derivatives of the functions

1. $y=\sin^2 x \quad \Rightarrow \quad y=(\sin x)^2 \quad \Rightarrow \quad \dfrac{dy}{dx}=2\sin x\cos x=\sin 2x$

2. $y=\cos(x^2) \quad \Rightarrow \quad \dfrac{dy}{dx}=-2x\sin(x^2)$

3. $y=\tan\sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{dx}=\sec^2\sqrt{x}\times\dfrac{1}{2\sqrt{x}}=\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

4. $y=x^2\sec 3x \quad \Rightarrow \quad \dfrac{dy}{dx}=3x^2\sec 3x\tan 3x+2x\sec 3x=x\sec 3x\,(2+3x\tan 3x)$

5. $y=\sqrt{\sin 2x} \quad \Rightarrow \quad y=(\sin 2x)^{1/2} \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{2}(\sin 2x)^{-1/2}\times\cos 2x\times 2$

$$=\frac{\cos 2x}{\sqrt{\sin 2x}}$$

Example 2: If $y=\tan 2t$ and $x=\sec 2t$ show that $\dfrac{dy}{dx}=\csc 2t$

$\dfrac{dy}{dt}=2\sec^2 2t \quad , \quad \dfrac{dx}{dt}=2\sec 2t\tan 2t$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=2\sec^2 2t\times\frac{1}{2\sec 2t\tan 2t}=\frac{\sec 2t}{\tan 2t}$$

$$=\frac{\frac{1}{\cos 2t}}{\frac{\sin 2t}{\cos 2t}}=\frac{1}{\sin 2t}=\csc 2t$$

Example3: If $y=\theta-\cos\theta$ and $x=\theta+\cos\theta\,;\left(0\le\theta\le\dfrac{\pi}{2}\right)$ show that $\dfrac{dy}{dx}=(\sec\theta+\tan\theta)^2$

$$\frac{dy}{d\theta}=1+\sin\theta \quad \text{and} \quad \frac{dx}{d\theta}=1-\sin\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{1+\sin\theta}{1-\sin\theta}$$

$$\frac{dy}{dx}=\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}=\frac{1+2\sin\theta+\sin^2\theta}{1-\sin^2\theta}=\frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$$

$$\frac{dy}{dx}=\frac{1}{\cos^2\theta}+\frac{2\sin\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\sec^2\theta+2\sec\theta\tan\theta+\tan^2\theta$$

$\therefore \quad \dfrac{dy}{dx}=(\sec\theta+\tan\theta)^2$

Inverse trigonometric functions

The inverse trigonometric functions are defined to be the inverses of particular parts of the trigonometric functions; parts that do have inverses.

The inverse sine function, denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function.

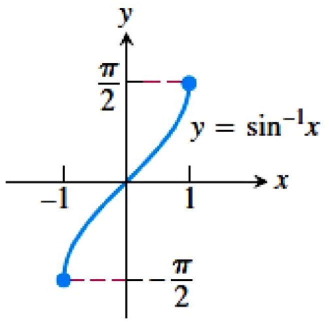
A similar idea holds for all the other inverse trigonometric functions. It is important here to note that in this case the (-1) is not an exponent and so,

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

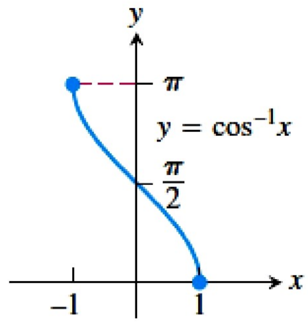
In inverse trigonometric functions the (-1) looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trigonometric function. It is a notation that we use in this case to denote inverse trigonometric functions. If we had really wanted exponentiation to denote 1 over sine we would use the following.

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

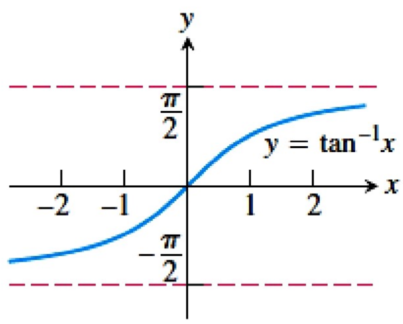
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



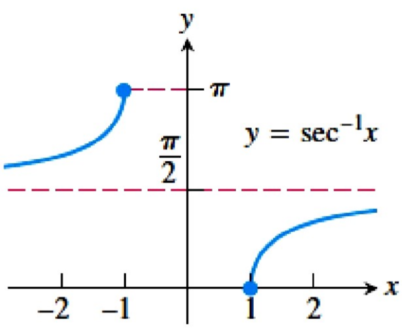
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



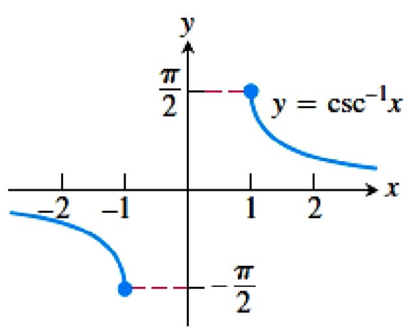
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



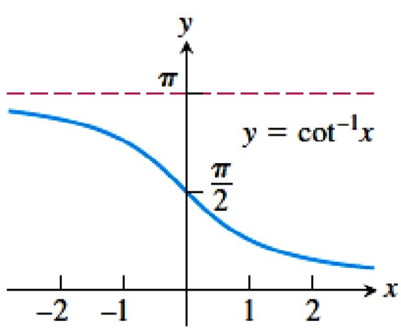
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \neq \frac{\pi}{2}$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \neq 0$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



Derivatives of inverse trigonometric functions

Let u be a function x, the derivatives of inverse trigonometric functions are:

1. $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} . \frac{du}{dx}$
2. $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} . \frac{du}{dx}$
3. $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} . \frac{du}{dx}$
4. $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} . \frac{du}{dx}$
5. $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} . \frac{du}{dx}$
6. $\frac{d}{dx}(\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} . \frac{du}{dx}$

Example 4: Find the derivative for

1. $y = \sin^{-1} 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \times 2 = \frac{2}{\sqrt{1-4x^2}}$

2. $y = 3x \cos^{-1} 3x - \sqrt{1-9x^2}$

$$\frac{dy}{dx} = 3x \times \frac{-1}{\sqrt{1-(3x)^2}} \times 3 + 3\cos^{-1} 3x - \frac{-18x}{2\sqrt{1-9x^2}}$$

$$= \frac{-9x}{\sqrt{1-9x^2}} + 3\cos^{-1} 3x + \frac{9x}{\sqrt{1-9x^2}} = 3\cos^{-1} 3x$$

3. $y = 2\sqrt{x} \tan^{-1} \sqrt{x}$

$$\frac{dy}{dx} = 2\sqrt{x} \times \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} + 2\tan^{-1} \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{1+x} + \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}$$

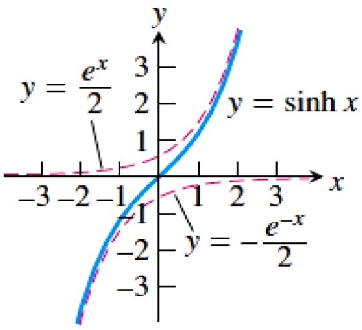
Exercises

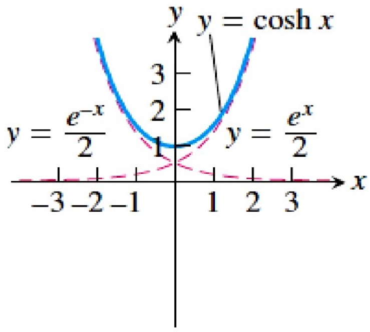
Find derivative in each of the following problems $(1-4)$

1. $y = \sec^2 2x$
2. $y = x^2 \sin x + 2x \cos x - 2\sin x$
3. $y = \sqrt{x^2-1} - \sec^{-1} x$
4. $y = 2x \cos^{-1} \sqrt{x} + \sin^{-1} \sqrt{x} - 2\sqrt{x-x^2}$
5. If $y = 1 - \sin\theta$ and $x = \theta - \sin\theta$ find $\frac{dy}{dx}$
6. If $y = \sec^{-1} t$ and $x = \sqrt{t^2-1}$ find $\frac{dy}{dx}$

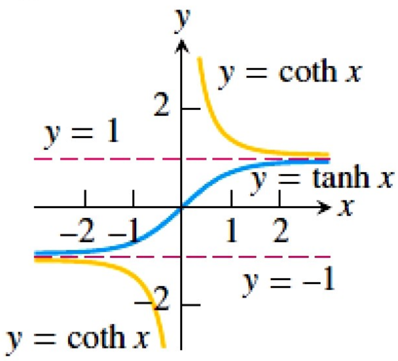
Hyperbolic functions الدوال الزائدية

نعلم بأن الدوال المثلثية (نسبة للمثلث) أو الدائرية (نسبة للدائرة) وهي الدوال $\sin x$, $\cos x$ واللتان تمثلان بإحداثيات النقطة $(\cos\theta ,\sin\theta)$ على محيط دائرة الوحدة والتي معادلتها $x^2+y^2=1$ حيث $y=\sin\theta$, $x=\cos\theta$. في حين ان الدوال الزائدية $\sinh x$, $\cosh x$ تحقق نقطة على منحنى القطع الزائد $x^2-y^2=1$ ومن هنا سميت بالدوال الزائدية.

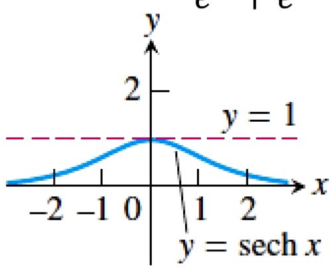


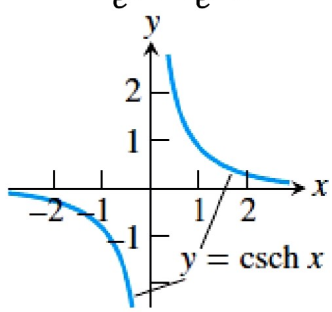


$$\sinh x=\frac{e^{x}-e^{-x}}{2}\qquad\qquad \cosh x=\frac{e^{x}+e^{-x}}{2}$$



$$\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\qquad\qquad \coth x=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$





$$\operatorname{sech} x=\frac{2}{e^{x}+e^{-x}}\qquad\qquad \operatorname{csch} x=\frac{2}{e^{x}-e^{-x}}$$

بعض المتطابقات

1. $\cosh x + \sinh x = e^{x}$

2. $\cosh x - \sinh x = e^{-x}$

3. $\cosh^2 x - \sinh^2 x = 1$

4. $\cosh(-x) = \cosh x \;\; \& \;\; \sinh(-x) = -\sinh x$

5. $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$

6. $\sinh 2x = 2\sinh x \cosh x$

7. $\cosh(x+y) = \cosh x \cosh y + \sinh y \sinh x$

8. $\cosh 2x = \cosh^2 x + \sinh^2 x$

9. $2\cosh^2 x = \cosh 2x + 1$

10. $2\sinh^2 x = \cosh 2x - 1$

البرهان

1. $\cosh x + \sinh x = \dfrac{e^{x}+e^{-x}}{2} + \dfrac{e^{x}-e^{-x}}{2} = e^{x}$

5. $\sinh x \cosh y + \sinh y \cosh x$

$$= \frac{e^{x}-e^{-x}}{2} \times \frac{e^{y}+e^{-y}}{2} + \frac{e^{y}-e^{-y}}{2} \times \frac{e^{x}+e^{-x}}{2}$$

$$= \frac{e^{x+y}+e^{x-y}-e^{-x+y}+e^{-x-y}}{4} + \frac{e^{y+x}+e^{y-x}-e^{-y+x}+e^{-y-x}}{4}$$

$$= \frac{2e^{x+y}+2e^{-x-y}}{4} = \frac{e^{x+y}+e^{-(x+y)}}{2} = \sinh(x+y)$$

8. $\cosh^2 x + \sinh^2 x = \left(\dfrac{e^{x}+e^{-x}}{2}\right)^2 + \left(\dfrac{e^{x}-e^{-x}}{2}\right)^2$

$$= \frac{1}{4}\left(e^{2x}+2+e^{-2x}+e^{2x}-2+e^{-2x}\right)$$

$$= \frac{e^{2x}+e^{-2x}}{2} = \cosh 2x$$

وبنفس الاسلوب نبرهن المتطابقات الأُخر .

مثال (١) بين ان $\lim_{x\to\infty} \tanh x = 1$

الحل :

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \lim_{x\to\infty} \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})}$$

$$= \lim_{x\to\infty} \frac{(1 - e^{-2x})}{(1 + e^{-2x})} = 1$$

مثال (٢) اثبت ان $\sinh 3x = 3\sinh x + 4\sinh^3 x$

الحل :

$$3\sinh x + 4\sinh^3 x = \frac{3}{2}(e^x - e^{-x}) + \frac{4}{8}(e^x - e^{-x})^3$$

$$= \frac{3}{2}(e^x - e^{-x}) + \frac{1}{2}[(e^x)^3 - 3(e^x)^2 e^{-x} + 3e^x(e^{-x})^2 - (e^{-x})^3]$$

$$= \frac{3}{2}(e^x - e^{-x}) + \frac{1}{2}(e^{3x} - 3e^x + 3e^{-x} - e^{-3x})$$

$$= \frac{3}{2}(e^x - e^{-x}) - \frac{3}{2}(e^x - e^{-x}) + \frac{1}{2}(e^{3x} - e^{-3x})$$

$$= \frac{1}{2}(e^{3x} - e^{-3x}) = \sinh 3x$$

مثال (٣) جد قيمة x اذا كان $\operatorname{sech} 2x = 0.25$

الحل:

$$\frac{2}{e^{2x} + e^{-2x}} = \frac{1}{4}$$

$$\{\ e^{2x} + e^{-2x} = 8 \quad \} \times e^{2x}$$

$$e^{4x} + 1 = 8e^{2x} \quad \rightarrow \quad e^{4x} - 8e^{2x} + 1 = 0$$

$$e^{2x} = \frac{8 \mp \sqrt{64 - 4}}{2 \times 1} = 4 \mp \sqrt{15}$$

$$2x = \ln\left(4 \mp \sqrt{15}\right) \quad \rightarrow \quad x = \frac{1}{2}\ln\left(4 \mp \sqrt{15}\right)$$

مثال (٤) حل المعادلة التالية $2\cosh 2x + 10\sinh 2x = 5$

الحل :

$$2 \times \frac{e^{2x}+e^{-2x}}{2} + 10 \times \frac{e^{2x}-e^{-2x}}{2} = 5$$

$$e^{2x}+e^{-2x}+5e^{2x}-5e^{-2x} = 5$$

$$\{ \quad 6e^{2x}-4e^{-2x} = 5 \quad \} \times e^{2x}$$

$$6e^{4x}-56e^{2x}-4=0$$

$$(2e^{2x}+1)(3e^{2x}-4)=0$$

$e^{2x} = -\frac{1}{2}$ تهمل

$$e^{2x} = \frac{4}{3} \quad \rightarrow \quad 2x = \ln\left(\frac{4}{3}\right) \qquad \therefore x = \frac{1}{2}\ln\left(\frac{4}{3}\right)$$

مشتقات الدوال الزائدية :

1. $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

2. $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$

3. $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$

4. $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$

5. $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$

6. $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$

البرهان

1. $\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u-e^{-u}}{2}\right) = \frac{1}{2}\left(e^u\frac{du}{dx}+e^{-u}\frac{du}{dx}\right)$

$$= \frac{1}{2}(e^u+e^{-u})\frac{du}{dx} = \cosh u \frac{du}{dx}$$

3. $\frac{d}{dx}(\tanh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{e^u + e^{-u}}\right)$

$$= \left(\frac{(e^u + e^{-u})(e^u + e^{-u}) - (e^u - e^{-u})(e^u + e^{-u})}{(e^u + e^{-u})^2}\right)\frac{du}{dx}$$

$$= \left(\frac{(e^{2u} + 2 + e^{-2u}) - (e^{2u} - 2 + e^{-2u})}{(e^u + e^{-u})^2}\right)\frac{du}{dx}$$

$$= \frac{4}{(e^u + e^{-u})^2}\frac{du}{dx} = \left(\frac{2}{e^u + e^{-u}}\right)^2\frac{du}{dx} = \operatorname{sech}^2 u\,\frac{du}{dx}$$

وبنفس الاسلوب نبرهن المشتقات الأُخر .

مثال (٥) جد معادلة المماس للمنحني $y = 3\cosh 2x - \sinh x$ عند $x = \ln 2$

الحل : $y' = 6\sinh 2x - \cosh x$

$$m = y'|_{x=\ln 2} = 6\sinh 2\ln 2 - \cosh\ln 2$$

$$m = 6\sinh\ln 4 - \cosh\ln 2 = 6 \times \frac{e^{\ln 4} - e^{-\ln 4}}{2} - \frac{e^{\ln 2} + e^{-\ln 2}}{2}$$

$$m = 3\left(4 - \frac{1}{4}\right) - \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{45}{4} - \frac{6}{4} = \frac{39}{4}$$

$$y|_{x=\ln 2} = 3\cosh\ln 4 - \sinh\ln 2$$

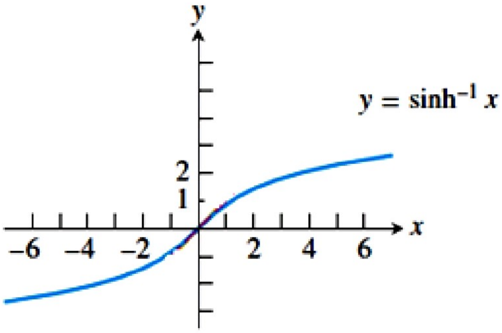
$$= \frac{3}{2}\left(4 + \frac{1}{4}\right) - \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{51}{8} - \frac{5}{4} = \frac{41}{8}$$

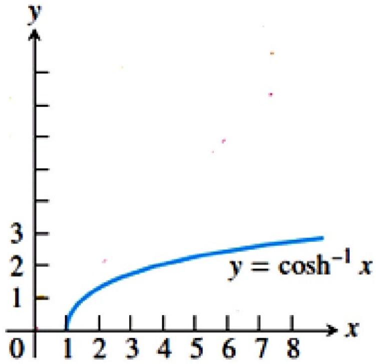
نقطة التماس $\left(\ln 2, \frac{41}{8}\right)$

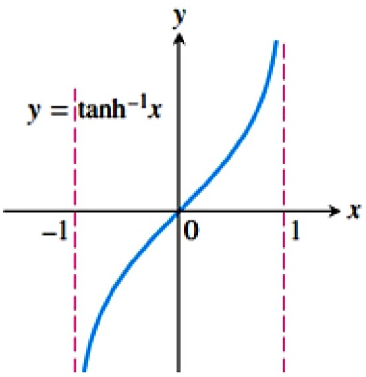
معادلة المماس $y - \frac{41}{8} = \frac{39}{4}(x - \ln 2)$

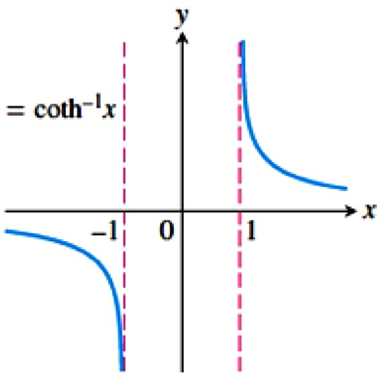
$$8y - 41 = 78(x - \ln 2)$$

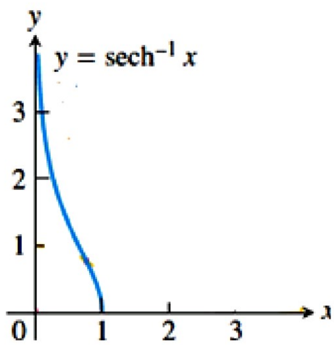
The inverse hyperbolic functions الدوال الزائدية العكسية

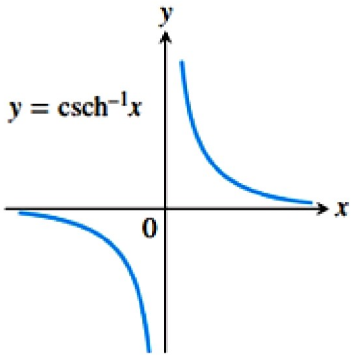












علاقات الدوال الزائدية العكسية بالدالة اللوغاريتمية

1. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad ; \quad -\infty < x < \infty$

2. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad ; \quad x \geq 1$

3. $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right) = \cosh^{-1}\left(\dfrac{1}{x}\right) \quad ; \; 0 < x \leq 1$

4. $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1 + x^2}}{|x|}\right) = \sinh^{-1}\left(\dfrac{1}{x}\right) \quad ; \quad x \neq 0$

5. $\coth^{-1} x = \dfrac{1}{2}\ln\dfrac{x + 1}{x - 1} = \tanh^{-1}\left(\dfrac{1}{x}\right) \quad ; \quad |x| \geq 1$

البرهان :

نفرض ان $1.\quad u = \sinh^{-1} x$

$$x = \sinh u = \frac{e^u - e^{-u}}{2}$$

$$\{\ 2x = e^u - e^{-u}\ \} \times e^u$$

$$e^{2u} - 2xe^u - 1 = 0$$

$$e^u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$e^u = x \pm \sqrt{x^2 + 1}$$

$$u = \ln\left(x \pm \sqrt{x^2 + 1}\right)$$

هنا نهمل القيمة $\ln\left(x - \sqrt{x^2+1}\right)$ لان $\forall x \quad x - \sqrt{x^2+1} < 0$

$$\therefore \quad \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

وبنفس الاسلوب نبرهن العلاقات الأُخر .

مشتقات الدوال الزائدية العكسية :

1. $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}} . \frac{du}{dx}$

2. $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} . \frac{du}{dx}$

3. $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2} . \frac{du}{dx} \quad ; |u| < 1$

4. $\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2} . \frac{du}{dx} \quad ; |u| > 1$

5. $\frac{d}{dx}(\text{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}} . \frac{du}{dx}$

6. $\frac{d}{dx}(\text{csch}^{-1} u) = \frac{-1}{u\sqrt{1+u^2}} . \frac{du}{dx}$

تمارين

1. بين ان $\lim_{x\to\infty} \coth x = -1$.

2. اثبت ان $\cosh 3x = 4\cosh^3 x - 3\cosh x$

3. جد مجموعة حل المعادلات التالية :

$$(a)\quad \cosh(\ln x) - \sinh\left(\ln \frac{1}{2} x\right) = \frac{7}{4}$$

$(b)\ 4\sinh x + 3e^x + 3 = 0$

$(c)\ 4\cosh x + \sinh x = 4$

$(d)\ 3\sinh x - \cosh x = 1$

$(e)\ 4\tanh x - \operatorname{sech} x = 1$

4. جد معادلة المماس للمنحني $y = 2\cosh x - 4\sinh x$ عند $x = \ln 3$.

5. جد مشتقة الدوال التالية :

$(a)\ y = \cosh 2x - \sinh 3x$

$(b)\ y = 4\sinh^{-1} 2x$

$(c)\ y = 5x \operatorname{sech} 4x - 21\tanh^3 4x$

$(d)\ y = 3\cosh^2 2x - 13\sinh^2(3x^2)$